ON THE CLASSIFICATION OF MULTI-ISOMETRIES

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Abstract. We consider the classification, up to unitary equivalence, of commuting n-tuples $(V_1, V_2, \ldots, V_n)$ of isometries on a Hilbert space. As in earlier work by Berger, Coburn, and Lebow, we start by analyzing the Wold decomposition of $V = V_1V_2\cdots V_n$, but unlike their work, we pay special attention to the case when $\ker V^*$ is of finite dimension. We give a complete classification of n-tuples for which V is a pure isometry of multiplicity n. It is hoped that deeper analysis will provide a classification whenever V has finite multiplicity. Further, we identify a pivotal operator in the case $n = 2$ which captures many of the properties of a bi-isometry.

The present work was initiated in 1999, during the memorial conference in honor of Béla Sz.-Nagy. This paper is a token of his perennial presence in mathematics.

1. Introduction

Much work in operator theory, particularly the model theory [16, 13] of B. Sz.-Nagy and the third author, relies on a good understanding and classification of isometric operators on Hilbert space. This understanding was initiated by J. von Neumann in a foundational paper on operator theory [17] where he demonstrates the decomposition of an isometry into a direct sum of a unitary and a unilateral shift. This decomposition was later rediscovered by H. Wold [18], who made it the cornerstone of prediction theory for stationary random processes. The deep relationship between harmonic analysis and shifts on Hilbert space was discovered then by A. Beurling [3]. The explosive development of operator theory and harmonic analysis which followed from these discoveries and Sz.-Nagy's dilation theory [14, 15] continues to this day.

By contrast, the understanding and classification of commuting pairs or, more generally, commuting families of isometries is very partial. A set of unitary invariants for finite sets of commuting isometries was found by C. Berger, L. Coburn and A. Lebow [2]; these invariants will also be considered in Section 2 below in somewhat more detail. Berger, Coburn and Lebow use these invariants in demonstrating that there are infinitely many nonisomorphic C^*-algebras generated by pairs of commuting isometries. Other authors [1, 4, 5, 9, 10] also demonstrate the great variety of families of commuting isometries, thus showing how difficult the classification problem can be.

In this note we review the unitary invariants of n-tuples of commuting isometries, originally introduced in [2] (cf. Theorems 2.1 and 2.8), we produce a smaller set of invariants (Theorem 2.3), and we determine some of the necessary conditions these invariants must satisfy (Proposition 2.4 and Theorem 2.5). The most specific results in this section pertain to the case $n = 3$. In this case, the possibility of finding triples of commuting isometries is related with the existence of invariant subspaces

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for a specific contraction (cf. Theorem 2.5 and Corollary 2.6). It is difficult to conjecture the natural extension of these results to $n \geq 4$. In Section 3 we obtain a deeper understanding of bi-isometries relating their structure to the model theory for contractions. Finally, in Section 4, we provide a complete classification for a certain class of irreducible n-tuples of commuting isometries. It is interesting to note that the proof of this classification result does not involve the unitary invariants of the n-tuples obtained earlier. While the possible unitary invariants can be explicitly calculated, at least for $n = 2$ and $n = 3$, the corresponding isometries are difficult to identify.

2. Model Multi-Isometries

We begin with a few general remarks about sets of commuting isometries. Consider a Hilbert space $\mathfrak{H}$, and a commutative semigroup $\mathfrak{S}$ of isometric operators on $\mathfrak{H}$; i.e., $VW = WV \in \mathfrak{S}$ whenever $V, W \in \mathfrak{S}$. As in the Wold-von Neumann decomposition, it was noted in [12] that the subspace

$$\mathfrak{H}_u = \bigcap_{V \in \mathfrak{S}} V\mathfrak{H}$$

is a reducing subspace for all the isometries in $\mathfrak{S}$ on which the restrictions to it are unitary. The semigroup $\mathfrak{S}$ is said to be *completely nonunitary* (or cnu) if $\mathfrak{H}_u = \{0\}$. Observe that individual elements of $\mathfrak{S}$ might have unitary parts even if $\mathfrak{S}$ is cnu.

If the semigroup $\mathfrak{S}$ is generated by a set $\mathfrak{G}$ of commuting isometries, we will also call $\mathfrak{H}_u$ the unitary part of $\mathfrak{G}$, and we will say that $\mathfrak{G}$ is cnu if $\mathfrak{S}$ is cnu. If $\mathfrak{G} = \{V_1, V_2, \dots, V_n\}$ is a finite set, observe that

$$\mathfrak{H}_u = \bigcap_{k=1}^{\infty} \bigcap_{i_1, i_2, \dots, i_k = 1}^{n} V_{i_1} V_{i_2} \cdots V_{i_k} \mathfrak{H} = \bigcap_{k=1}^{\infty} V^k \mathfrak{H},$$

where $V = V_1 V_2 \cdots V_n$; indeed, this is seen from the inclusion

$$V_{i_1} V_{i_2} \cdots V_{i_k} \mathfrak{H} \subset V^k \mathfrak{H}$$

Thus, as remarked earlier in [2], $\{V_1, V_2, \dots, V_n\}$ is cnu if and only if V is a unilateral shift (of arbitrary multiplicity).

For easier reference, an ordered n-tuple $(V_1, V_2, \dots, V_n)$ of commuting isometries will be called simply an n-isometry. In this paper we will focus mostly on completely nonunitary n-isometries. However, when the n-isometry $\{V_1, V_2, \dots, V_n\}$ is not completely non-unitary, then the system of restrictions $V_i|_{\mathcal{H}_u}$ of V_i, $i = 1, 2, \dots, n$, will be called the unitary part of the n-isometry, while the system of restrictions to $\mathcal{H} \ominus \mathcal{H}_u$ will be called the cnu part.

We recall now that every shift can be realized conveniently as an operator on a Hardy space. Thus, given a Hilbert space $\mathfrak{E}$, consider the Hardy space $H^2(\mathfrak{E})$ of Taylor series with square summable coefficients in $\mathfrak{E}$. The operator $S_{\mathfrak{E}}$ defined by

$$(S_{\mathfrak{E}} f)(z) = z f(z), \quad f \in H^2(\mathfrak{E}), \quad z \in \mathbb{D} = \{\lambda \in \mathbb{C} \colon \ |\lambda| < 1\},$$

is a unilateral shift of multiplicity equal to the dimension of $\mathfrak{E}$. It will be convenient to identify $\mathfrak{E}$ with the collection of constant functions in $H^2(\mathfrak{E})$.

In view of the above considerations, it is of interest to study isometries V for which $S_{\mathfrak{E}} = VW$, where W is some other isometry commuting with V. We will call such an isometry V an isometric divisor of $S_{\mathfrak{E}}$. The following description of divisors was first proved in [2]. Our proof is somewhat simpler.

For P a projection, we set $P^{\perp} = I - P$.

Theorem 2.1. *The following are equivalent.*
(1) *V is an isometric divisor of $S_{\mathfrak{E}}$.*
(2) *There exist a unitary operator U on $\mathfrak{E}$ and an orthogonal projection P on $\mathfrak{E}$ such that $(Vf)(z) = U(zP + P^{\perp})f(z)$ for all $f \in H^2(\mathfrak{E})$.*

Proof. It is trivial to verify that an operator V, as given in (2), is a divisor of $S_{\mathfrak{E}}$ and, in fact, $VW = WV = S_{\mathfrak{E}}$, with $(Wf)(z) = (P + zP^{\perp})U^* f(z)$. Conversely, assume W is an isometry and $VW = WV = S_{\mathfrak{E}}$. Since V and W commute with $S_{\mathfrak{E}}$, there exist inner $\mathfrak{L}(\mathfrak{E})$-valued functions Θ, Ω such that $\Theta(z)\Omega(z) = \Omega(z)\Theta(z) = zI_{\mathfrak{E}}$, $(Vf)(z) = \Theta(z)f(z)$, and $(Wf)(z) = \Omega(z)f(z)$ for $f \in H^2(\mathfrak{E})$ and $|z| < 1$. Now, the range of W contains the range of $S_{\mathfrak{E}}$, so there exists a projection P on $\mathfrak{E}$ for which

$$WH^2(\mathfrak{E}) = S_{\mathfrak{E}}H^2(\mathfrak{E}) \oplus P\mathfrak{E} = \Omega_1 H^2(\mathfrak{E}),$$

where $\Omega_1(z) = zP^{\perp} + P$. The Beurling-Lax-Halmos theorem provides a unitary operator U on $\mathfrak{E}$ such that $\Omega_1(z) = \Omega(z)U$ for $|z| < 1$. It is easy now to conclude for almost every z on the unit circle that

$$\Theta(z) = z\Omega(z)^* = zU\Omega_1(z)^* = U(P^{\perp} + zP),$$

so that V satisfies (2). □

We will denote by $V_{U,P}$ the divisor of $S_{\mathfrak{E}}$ described in condition (2) of the above result.

Lemma 2.2. *Consider unitary operators U, U_1, U_2, and orthogonal projections P, P_1, P_2 on the Hilbert space $\mathfrak{E}$. The following are equivalent.*
(1) *$V_{U,P} = V_{U_1,P_1}V_{U_2,P_2}$;*
(2) *$U = U_1U_2$, and $P = P_2 + U_2^* P_1 U_2$.*

Proof. We only argue the nontrivial implication (1)⇒(2). Identification of coefficients yields the equations $U_1P_1U_2P_2 = 0$, $U_1[P_1U_2P_2^{\perp} + P_1^{\perp}U_2P_2] = UP$, and $U_1P_1^{\perp}U_2P_2^{\perp} = UP^{\perp}$. The first two relations yield $U_1P_1U_2 + U_1U_2P_2 = UP$, while the first and third yield $U_1P_1^{\perp}U_2 - V_1V_2P_2 = VP^{\perp}$, where $V = V_{U,P}$. Adding these last two equalities we obtain $U = U_1U_2$, and therefore $U_1U_2P = VP = U_1P_1U_2 + U_1U_2P_2$. This gives immediately the second equality in (2). □

It should be noted that the relation $P = P_2 + U_2^* P_1 U_2$ contains implicitly the fact that the two projections on the right hand side are orthogonal or, equivalently, $P_1U_2P_2 = 0$. Indeed, the sum $Q = Q_1 + Q_2$ of two orthogonal projections Q_1, Q_2 is a contraction if and only if the ranges of Q_1 and Q_2 are orthogonal, and in this latter case Q is the orthogonal projection onto the span of the two ranges.

We can now give a model for arbitrary cnu n-isometries. Consider a Hilbert space $\mathfrak{E}$, unitary operators $U_1, U_2, \ldots, U_n$ on $\mathfrak{E}$, and orthogonal projections $P_1, P_2, \ldots, P_n$ on $\mathfrak{E}$. The n-tuple $(V_{U_1,P_1}, V_{U_2,P_2}, \ldots, V_{U_n,P_n})$ is called a *model n-isometry* if the following conditions are satisfied:

(a) $U_iU_j = U_jU_i$ for $i, j = 1, 2, \ldots, n$;
(b) $U_1U_2 \cdots U_n = I_{\mathfrak{E}}$;
(c) $P_j + U_j^* P_i U_j = P_i + U_i^* P_j U_i \leq I_{\mathfrak{E}}$ for $i \neq j$; and
(d) $P_1 + U_1^* P_2 U_1 + U_1^* U_2^* P_3 U_2 U_1 + \cdots + U_1^* U_2^* \cdots U_{n-1}^* P_n U_{n-1} \cdots U_2 U_1 = I_{\mathfrak{E}}$.

Observe again that the projections appearing in (d) must be pairwise orthogonal. It follows from an inductive application of the preceding lemma that a model n-isometry is indeed an n-isometry, and

$$V_{U_1,P_1}V_{U_2,P_2}\cdots V_{U_n,P_n}=S_{\mathfrak{E}}.$$

When $n=2$ we have $U_2=U_1^*$ and $P_2=I-U_1P_1U_1^*$, so a model two-isometry is determined by U_1 and P_1. More generally, a model n-isometry is determined by U_j,P_j for $1\le j\le n-1$. The relevant conditions on these operators are as follows.

Theorem 2.3. *Let $\mathfrak{E}$ be a Hilbert space, $(U_j)_{j=1}^{n-1}$ unitary operators on $\mathfrak{E}$, and $(P_j)_{j=1}^{n-1}$ orthogonal projections on $\mathfrak{E}$. Assume that the following conditions are satisfied:*

(1) *$U_iU_j=U_jU_i$ for $i,j=1,2,\dots,n-1$;*
(2) *$P_j+U_j^*P_iU_j=P_i+U_i^*P_jU_i\le I_{\mathfrak{E}}$ for $i\ne j$, $1\le i,j\le n-1$; and*
(3) *$P_1+U_1^*P_2U_1+\cdots+U_1^*U_2^*\cdots U_{n-2}^*P_{n-1}U_{n-2}\cdots U_2U_1\le I_{\mathfrak{E}}$.*

Then there exists a unique unitary operator U_n on $\mathfrak{E}$ and a unique projection P_n on $\mathfrak{E}$, such that $(V_{U_j,P_j})_{j=1}^n$ is a model n-isometry.

Proof. Observe that the projections in condition (3) must be pairwise orthogonal. An inductive application of the above lemma shows that the product

$$V_{U_1,P_1}V_{U_2,P_2}\cdots V_{U_{n-1},P_{n-1}}$$

is equal to $V_{U,P}$, where $U=U_1U_2\cdots U_{n-1}$, and P is the sum in the left hand side of condition (3). We can then choose $U_n=U^*$ and $P_n=I-UPU^*$ to obtain a model n-isometry, and this is clearly the only possible choice. $\square$

The conditions in this proposition are vacuously satisfied if $n=2$, but are fairly stringent for larger n. We illustrate this in case $n=3$, when the only conditions on (U_1,U_2,P_1,P_2) are that $U_1U_2=U_2U_1$ and

$$P_2+U_2^*P_1U_2=P_1+U_1^*P_2U_1\le I_{\mathfrak{E}}.$$

Observe that the data (U_1,U_2,P_1,P_2) can be recovered from (U_1,U_2,P_1,Q_1), where $Q_1=U_1^*P_2U_1$, and the required conditions can be written more easily in terms of the mutually orthogonal projections P_1 and Q_1. Moreover, as will be seen below, the range of Q_1 is an invariant subspace for a suitably defined compression of V_{U_1,P_1}.

Proposition 2.4. *Consider unitary operators U_1,U_2, and orthogonal projections P_1,Q_1 on a Hilbert space $\mathfrak{E}$ such that $U_1U_2=U_2U_1$ and $P_1+Q_1\le I_{\mathfrak{E}}$. Let us also set*

$$P_2=U_1Q_1U_1^*, Q_2=U_2^*P_1U_2, U=U_1U_2, \text{ and } T_1=P_1^\perp U_1|P_1^\perp\mathfrak{E}.$$

(1) *We have $P_2\le P_1+Q_1$ if and only if $Q_1\mathfrak{E}$ is an invariant subspace for T_1.*
(2) *We have $Q_2\le P_1+Q_1$ if and only if $Q_1\mathfrak{E}$ contains $P_1^\perp U_2^*P_1\mathfrak{E}$.*
(3) *We have $P_2Q_2=0$ if and only if $Q_1\mathfrak{E}$ is contained in $U^*P_1^\perp\mathfrak{E}$.*

In summary, the inequalities

$$P_2+U_2^*P_1U_2\le P_1+U_1^*P_2U_1\le I_{\mathfrak{E}}$$

hold if and only if $Q_1\mathfrak{E}$ is an invariant subspace for T_1 such that

$$P_1^\perp U_2^*P_1\mathfrak{E}\subset Q_1\mathfrak{E}\subset U^*P_1^\perp\mathfrak{E}.$$

In particular, if these inclusions hold, we have

$$P_1U[P_1^\perp U_1]^nP_1^\perp U_2^*P_1=0\ \textit{for}\ n\ge 0.$$

Proof. Note that $P_2 \leq P_1 + Q_1$ is equivalent to the inclusion $P_1^\perp P_2\mathfrak{E} \subset Q_1\mathfrak{E}$. Now (1) follows immediately since $P_1^\perp P_2\mathfrak{E} = P_1^\perp U_1 Q_1\mathfrak{E}$. Similarly, $Q_2 \leq P_1 + Q_1$ is equivalent to $P_1^\perp Q_2\mathfrak{E} \subset Q_1\mathfrak{E}$, which renders (2) obvious. Finally, $P_2Q_2 = 0$ if and only if $P_2\mathfrak{E} \subset Q_2^\perp\mathfrak{E}$ or, equivalently,

$$Q_1\mathfrak{E} = U_1^* P_2\mathfrak{E} \subset U_1^* Q_2^\perp\mathfrak{E}.$$

The inclusion in (3) follows because

$$U_1^* Q_2^\perp\mathfrak{E} = U_1^* Q_2^\perp U_1\mathfrak{E} = (I - U_1^* Q_2 U_1)\mathfrak{E} = (I - U^* P_1 U)\mathfrak{E} = U^* P_1^\perp\mathfrak{E}.$$

For the last assertion in the statement, notice first that the condition $P_2 + Q_2 \leq I_\mathfrak{E}$ implies $P_2Q_2 = 0$. Therefore, this statement simply says that the invariant subspace for T_1, generated by $P_1^\perp U_2^* P_1\mathfrak{E}$, is contained in $U^* P_1^\perp\mathfrak{E}$. □

The preceding result shows the importance of the operator T_1 which we call the "povital operator" and consider further in the following section. Moreover, the result suggests introducing the spaces

$$\mathfrak{Q}_{1,\min} = \bigvee_{k=0}^{\infty} T_1^k P_1^\perp U_2^* P_1\mathfrak{E},$$

and

$$\mathfrak{Q}_{1,\max} = P_1^\perp\mathfrak{E} \ominus \bigvee_{k=0}^{\infty} T_1^{*k} P_1^\perp U_1^* U_2^* P_1\mathfrak{E}.$$

In other words, $\mathfrak{Q}_{1,\min}$ is the smallest invariant subspace for T_1 containing $P_1^\perp U_2^* P_2\mathfrak{E}$, while $\mathfrak{Q}_{1,\max}$ is the largest invariant subspace for T_1 contained in $P_1^\perp\mathfrak{E} \ominus P_1^\perp U^* P_1\mathfrak{E}$.

Theorem 2.5. *Assume that $\mathfrak{E}$ is a Hilbert space, U_1, U_2 are commuting unitary operators on $\mathfrak{E}$ and P_1 is an orthogonal projection on $\mathfrak{E}$. There exists a projection P_2 on $\mathfrak{E}$ such that*

$$P_2 + U_2^* P_1 U_2 \leq P_1 + U_1^* P_2 U_1 \leq I_\mathfrak{E}$$

if and only if $\mathfrak{Q}_{1,\min} \subset \mathfrak{Q}_{1,\max}$. When this condition is satisfied, the general form of such projections P_2 is $P_2 = U_1 Q_1 U_1^*$, where Q_1 is the orthogonal projection onto an invariant subspace of T_1 satisfying

$$\mathfrak{Q}_{1,\min} \subset Q_1\mathfrak{E} \subset \mathfrak{Q}_{1,\max}.$$

Proof. This is just a summary of the preceding discussion. □

When $\mathfrak{E}$ is finite dimensional, the statement can be made more precise.

Corollary 2.6. *Under the hypotheses of the preceding theorem, assume that $\mathfrak{E}$ is finite dimensional and $\mathfrak{Q}_{1,\min} \subset \mathfrak{Q}_{1,\max}$.*

(1) *If Q_1 is an arbitrary projection onto an invariant subspace of T_1, and*

$$\mathfrak{Q}_{1,\min} \subset Q_1\mathfrak{E} \subset \mathfrak{Q}_{1,\max},$$

then

$$P_2 + U_2^* P_1 U_2 = P_1 + U_1^* P_2 U_1 \leq I_\mathfrak{E},$$

where $P_2 = U_1 Q_1 U_1^*$. Thus, there exists a 3-isometry of the form

$$(V_{U_1,P_1}, V_{U_2,P_2}, V_{U_3,P_3})$$

such that $V_{U_1,P_1} V_{U_2,P_2} V_{U_3,P_3} = S_\mathfrak{E}$.

(2) *The space $\mathfrak{Q}_{1,\min}$ is $\{0\}$ if and only if among the 3-isometries in (1) there is one where V_{U_2,P_2} is unitary.*

(3) *The space* $\mathfrak{Q}_{1,\max}$ *equals* $P_1^\perp \mathfrak{E}$ *if and only if among the 3-isometries in* (1) *there is one such that* V_{U_3,P_3} *is unitary.*

Proof. The last part of the first statement can be deduced from the fact that the projections $P_2+U_2^*P_1U_2$ and $P_1+U_1^*P_2U_1$ have the same rank, equal to the sum of the ranks of P_1 and P_2. The second statement follows from the fact that Q_1 can be chosen to be zero if and only if $\mathfrak{Q}_{1,\min}=\{0\}$. Likewise, the third statement follows from the fact that Q_1 can be chosen to be $P_1^\perp$ if and only if $\mathfrak{Q}_{1,\max}=P_1^\perp\mathfrak{E}$. □

One could ask whether part (1) of the preceding corollary is true when the dimension of $\mathfrak{E}$ is infinite. Unfortunately, the answer is negative, as shown by the following example.

Example 2.7. Denote by $\mathfrak{E}=L^2$ the usual space of square integrable functions on the unit circle (relative to normalized arclength measure), and define unitary operators U_1,U_2 on $\mathfrak{E}$ by setting

$$(U_1f)(z)=\varphi(z)f(z),(U_2f)(z)=zf(z),\quad f\in\mathfrak{E},z\in\partial\mathbb{D},$$

where the function φ is defined to be equal to 1 on the upper half-circle, and -1 on the lower half-circle. Consider also the orthogonal projection P_1 on $\mathfrak{E}$ such that $P_1\mathfrak{E}=(H^2)^\perp$. We claim that with these choices we have

$$\mathfrak{Q}_{1,\min}=\mathfrak{Q}_{1,\max}=\{0\},$$

and for the (unique) choice $Q_1=0$ we have

$$P_2+U_2^*P_1U_2\lneqq P_1+U_1^*P_2U_2.$$

Indeed, U_2 leaves $H^2\subset\mathfrak{E}$ invariant. Therefore, $(P_1^\perp)U_2^*P_1=0$, which in turn implies that $\mathfrak{Q}_{1,\min}=\{0\}$. On the other hand,

$$(P_1^\perp)\mathfrak{E}\ominus\mathfrak{Q}_{1,\max}=\bigvee_{n=0}^{\infty}T_1^{*n}(P_1^\perp)U_1^*U_2^*P_1\mathfrak{E}$$

is the smallest invariant subspace for the operator $T_1^*=T_1=T_\varphi$ generated by the space $(P_1^\perp)U_1^*U_2^*(H_2)^\perp=P_{H^2}[\overline{\varphi z}(H^2)^\perp]$. This space is actually dense in H^2 already. Indeed, consider a function $u\in H^2\ominus P_{H^2}[\overline{\varphi z}(H^2)^\perp]=H^2\cap(\varphi zH^2)$. There must exist $v\in H^2$ such that $u=\varphi zv$. The F. and M. Riesz theorem implies (by looking at the upper half-circle) that $u=zv$ and (looking at the lower half-circle) $u=-zy$, and therefore $u=0$. It is now easy to see that

$$P_2+U_2^*P_1U_2=U_2^*P_1U_2=P_{\overline{z}(H^2)^\perp}\neq P_1+U_1^*P_2U_1=P_1=P_{(H^2)^\perp}.$$

The discussion above shows that constructing cnu 3-isometries can be a delicate task. If $\mathfrak{Q}_{1,\max}\supset\mathfrak{Q}_{1,\min}$ and the compression of the pivotal operator T_1 in Theorem 2.5 to the space $\mathfrak{Q}_{1,\max}\ominus\mathfrak{Q}_{1,\min}$ is transitive (e.g., when this space has dimension zero or one), then the only choices for Q_1 are the projections onto $\mathfrak{Q}_{1,\max},\mathfrak{Q}_{1,\min}$, and even these may fail to produce 3-isometries, as seen in the preceding example. It is, however, possible to formulate equivalent conditions for the existence of P_2, given U_1,U_2 and P_1. To find these conditions assume that, given this data, the inclusion $\mathfrak{Q}_{1,\max}\supset\mathfrak{Q}_{1,\min}$ is satisfied. According to Theorem 2.5, the inequalities

$$P_2+U_2^*P_1U_2\leq P_1+U_1^*P_2U_1\leq I_{\mathfrak{E}}$$

are satisfied when $P_2 = U_1Q_1U_1^*$, and Q_1 is the orthogonal projection on either of the spaces $\mathfrak{Q}_{1,\max}, \mathfrak{Q}_{1,\min}$. This allows us to define an isometric operator

$$W : P_1 + \mathfrak{Q}_{1,\max} \to P_1 + \mathfrak{Q}_{1,\max}$$

by setting

$$W(x_1 + x_2) = U_2^* x_1 + U_1 x_2, \quad x_1 \in P_1\mathfrak{E}, x_2 \in \mathfrak{Q}_{1,\max}.$$

Theorem 2.8. *Assume that $\mathfrak{E}$ is a Hilbert space, U_1, U_2 are commuting unitary operators on $\mathfrak{E}$, P_1 is an orthogonal projection, and $\mathfrak{Q}_{1,\min} \subset \mathfrak{Q}_{1,\max}$. Define the isometry W as above and consider the operator T_1 used in Proposition 2.4.*

(1) *A subspace $\mathfrak{Q}_1$ such that $\mathfrak{Q}_{1,\min} \subset \mathfrak{Q}_1 \subset \mathfrak{Q}_{1,\max}$ is invariant for T_1 if and only if $P_1\mathfrak{E} + \mathfrak{Q}_1$ is invariant for W.*
(2) *Let $\mathfrak{Q}_1$ be an invariant subspace of T_1 such that $\mathfrak{Q}_{1,\min} \subset \mathfrak{Q}_1 \subset \mathfrak{Q}_{1,\max}$, denote by Q_1 the orthogonal projection onto $\mathfrak{Q}_1$, and set $P_2 = U_1Q_1U_1^*$. We have*
$$P_2 + U_2^*P_1U_2 = P_1 + U_1^*P_2U_1 \le I_{\mathfrak{E}}$$
if and only if $W|(P_1\mathfrak{E} + \mathfrak{Q}_1)$ is a unitary operator.
(3) *There exists an orthogonal projection P_2 on $\mathfrak{E}$ such that*
$$P_2 + U_2^*P_1U_2 = P_1 + U_1^*P_2U_1 \le I_{\mathfrak{E}}$$
if and only if $P_1\mathfrak{E} + \mathfrak{Q}_{1,\min}$ is contained in the unitary part of W in the von Neumann—Wold decomposition. The collection of such projections P_2 is a complete lattice.

Proof. Assume first that $\mathfrak{Q}_1$ is invariant for T_1. As noted above, W leaves $P_1\mathfrak{E} + \mathfrak{Q}_{1,\min}$ invariant. Therefore, for $p_1 \in P_1\mathfrak{E}$ and $q_1 \in \mathfrak{Q}_1$ we have

$$W(p_1 + q_1) = Wp_1 + P_1U_1q_1 + T_1q_1 \in P_1\mathfrak{E} + \mathfrak{Q}_{1,\min} + \mathfrak{Q}_1 \subset P_1\mathfrak{E} + \mathfrak{Q}_1.$$

Conversely, if W leaves $P_1\mathfrak{E} + \mathfrak{Q}_1$ invariant, the above formula can be rewritten as

$$T_1q_1 = W(p_1 + q_1) - Wp_1 \in P_1\mathfrak{E} + \mathfrak{Q}_1,$$

and this clearly implies that T_1 leaves $\mathfrak{Q}_1$ invariant. This proves (1). To verify (2), one only needs to observe that the range of $P_2 + U_2^*P_1U_2$ is precisely equal to $W(P_1\mathfrak{E} + \mathfrak{Q}_1)$. Part (3) follows immediately from (1) and (2). □

The preceding result also clarifies Example 2.7, for which the isometry W is pure.

The following result appears in a slightly different form as Theorem 3.2 in [2].

Theorem 2.9.

(1) *Any cnu n-isometry is unitarily equivalent to a model n-isometry.*
(2) *Consider two model n-isometries $(V_{U_j,P_j})_{j=1}^n$ and $(V_{U'_j,P'_j})_{j=1}^n$, where U_j, P_j act on $\mathfrak{E}$ and U'_j, P'_j act on $\mathfrak{E}'$. These model n-isometries are unitarily equivalent if and only if there exists a unitary operator $W : \mathfrak{E} \to \mathfrak{E}'$ satisfying $WU_j = U'_jW$ and $WP_j = P'_jW$ for all j.*

Proof. Let $(V_1, V_2, \dots, V_n)$ be a cnu n-isometry. Up to unitary equivalence, we may assume that $V_1V_2\cdots V_n = S_{\mathfrak{E}}$ for some Hilbert space $\mathfrak{E}$. It follows then that each V_j is of the form V_{U_j,P_j}, and properties (a)—(d) follow from the Lemma 2.2. The second part of the statement follows from the fact that any unitary operator $X : H^2(\mathfrak{E}) \to H^2(\mathfrak{E}')$ satisfying $XS_{\mathfrak{E}} = S_{\mathfrak{E}'}X$ must be the multiplication operator defined by some (constant) unitary $W : \mathfrak{E} \to \mathfrak{E}'$. □

It is natural to characterize various properties of n-isometries in terms of the corresponding models. This is thoroughly pursued in [7]. Here we only note the following result. We recall that operators T and S are said to doubly commute if $TS = ST$ and $T^*S = ST^*$.

Proposition 2.10. *A model two-isometry $(V_{U_1,P_1}, V_{U_2,P_2})$ consists of doubly commuting operators if and only if $P_1U_1^*P_1 = U_1^*P_1$, i.e., U_1^* leaves the range of P_1 invariant. Furthermore, these conditions are satisfied if and only the pivotal operator T_1 is an isometry.*

Proof. Fix $f \in H^2(\mathfrak{E})$ and $|z| < 1$. A calculation shows that

$$(V^*_{U_2,P_2}f)(z) = U_1P_1^\perp \frac{f(z) - f(0)}{z} + U_1P_1f(z),$$

and further computation yields

$$[(V^*_{U_2,P_2}V_{U_1,P_1} - V_{U_1,P_1}V^*_{U_2,P_2})f](z) = U_1P_1U_1P_1^\perp f(0).$$

Thus double commutation is equivalent to $P_1U_1P_1^\perp = 0$. □

A related result is observed in [2], namely that the commutator $V_1^*V_2 - V_2V_1^*$ is compact if and only if $P_1U_1P_1^\perp$ is compact.

The preceding result extends in the obvious way to arbitrary model n-isometries. The condition for double commutativity is simply that, for each j, U_j^* leaves the range of P_j invariant.

3. The Invariants of Bi-Isometries

As noted above, a complete set of unitary invariants of cnu bi-isometries is provided by triples $(\mathfrak{E}, U, P)$, where $\mathfrak{E}$ is a Hilbert space and U and P are operators on $\mathfrak{E}$ with U unitary and P an orthogonal projection. The model operators are V_{U_1,P_1} and V_{U_2,P_2}, where $U_1 = U$, $P_1 = P$, $U_2 = U^*$, and $P_1 = UP^\perp U^*$. For easier reference, we will call such a triple a model triple. Two model triples $(\mathfrak{E}, U, P)$ and $(\mathfrak{E}_1, U_1, P_1)$ determine unitarily equivalent bi-isometries if and only if there exists a unitary operator $A : \mathfrak{E} \to \mathfrak{E}_1$ satisfying $AU = U_1A$ and $AP = P_1A$.

Fix a model triple $(\mathfrak{E}, U, P)$, and introduce auxiliary spaces $\mathfrak{F} = P^\perp\mathfrak{E}$, $\mathfrak{D} = (\mathfrak{F} \vee U\mathfrak{F}) \ominus \mathfrak{F}$, $\mathfrak{F}' = \mathfrak{E} \ominus (\mathfrak{F} \vee U\mathfrak{F})$, $\mathfrak{D}_* = (\mathfrak{F} \vee U^*\mathfrak{F}) \ominus \mathfrak{F}$, and $\mathfrak{F}'_* = \mathfrak{E} \ominus (\mathfrak{F} \vee U^*\mathfrak{F})$. Observe that $\mathfrak{F} \vee U\mathfrak{F} = \mathfrak{F} \oplus \mathfrak{D}$, $\mathfrak{F} \vee U^*\mathfrak{F} = \mathfrak{F} \oplus \mathfrak{D}_*$, $\mathfrak{D} \oplus \mathfrak{F}' = \mathfrak{D}_* \oplus \mathfrak{F}'_*$, $U(\mathfrak{F} \vee U^*\mathfrak{F}) = \mathfrak{F} \vee U\mathfrak{F}$, and consequently $U\mathfrak{F}'_* = \mathfrak{F}$. Consider also the contraction operator T on $\mathfrak{F}$ defined by $T = P^\perp U|\mathfrak{F}$. (Note that T coincides with the pivotal operator T_1 in Proposition 2.4.) Our goal in this section is to understand better the relation of T to the corresponding bi-isometry.

It is well-known (cf. for example Theorem IV.3.1 in [6]) that the unitary operator $U|\mathfrak{F} \oplus \mathfrak{D}_* : \mathfrak{F} \oplus \mathfrak{D}_* \to \mathfrak{F} \oplus \mathfrak{D}$ is essentially given by the Julia-Halmos matrix

$$\begin{bmatrix} T & D_{T^*} \\ D_T & -T^* \end{bmatrix},$$

where $D_T = (I - T^*T)^{1/2}$ and $D_{T^*} = (I - TT^*)^{1/2}$. More precisely, there are unique unitary operators $W : \mathfrak{D}_T = (D_T\mathfrak{E})^- \to \mathfrak{D}$ and $W_* : \mathfrak{D}_{T^*} \to \mathfrak{D}_*$ such that, for $f \in \mathfrak{F}$ and $d_* \in \mathfrak{D}_*$ we have

$$U(f \oplus d_*) = [Tf + D_{T^*}W_*^*d_*] \oplus [WD_Tf - WT^*W_*^*d_*].$$

To simplify the notation, we replace the original model triple by the equivalent model triple $(\mathfrak{F}\oplus\mathfrak{D}_T\oplus\mathfrak{F}', \Omega^*U\Omega, \Omega^*P\Omega = P_{\mathfrak{F}\oplus\{0\}\oplus\{0\}})$, where $\Omega : \mathfrak{F}\oplus\mathfrak{D}_T\oplus\mathfrak{F}' \to \mathfrak{E}$ is given by $\Omega(f\oplus d\oplus f') = f + Wd + f'$. If we also consider the unitary Ω' : $\mathfrak{F}\oplus\mathfrak{D}_{T^*}\oplus\mathfrak{F}' \to \mathfrak{E}$ given by $\Omega'(f\oplus d_*\oplus f') = f + W_*d_* + U^*f'$, the new unitary $\Omega^*U\Omega$ can be factored as

$$\Omega^*U\Omega = (\Omega^*U\Omega')(\Omega'^*\Omega),$$

and now the operator $\Omega^*U\Omega' : \mathfrak{F}\oplus\mathfrak{D}_{T^*}\oplus\mathfrak{F}' \to \mathfrak{F}\oplus\mathfrak{D}_T\oplus\mathfrak{F}'$ is represented by the matrix

$$\Omega^*U\Omega' = \begin{bmatrix} T & D_{T^*} & 0 \\ D_T & -T^* & 0 \\ 0 & 0 & I_{\mathfrak{F}'} \end{bmatrix},$$

while $\Omega'^*\Omega : \mathfrak{F}\oplus\mathfrak{D}_T\oplus\mathfrak{F}' \to \mathfrak{F}\oplus\mathfrak{D}_{T^*}\oplus\mathfrak{F}'$ must have the form

$$\Omega'^*\Omega = \begin{bmatrix} I_{\mathfrak{F}} & 0 \\ 0 & Z \end{bmatrix},$$

where $Z : \mathfrak{D}_T\oplus\mathfrak{F}' \to \mathfrak{D}_{T^*}\oplus\mathfrak{F}'$ is a unitary. We summarize this construction in the following result.

Proposition 3.1. *Consider Hilbert spaces $\mathfrak{F}, \mathfrak{F}'$, a contraction T on $\mathfrak{F}$, and a unitary operator $Z : \mathfrak{D}_T\oplus\mathfrak{F}' \to \mathfrak{D}_{T^*}\oplus\mathfrak{F}'$. Associated with this data is a model triple $(\mathfrak{E}, U, P)$, where $\mathfrak{E} = \mathfrak{F}\oplus\mathfrak{D}_T\oplus\mathfrak{F}'$, $P^\perp\mathfrak{E} = \mathfrak{F}\oplus\{0\}\oplus\{0\}$, and $U = W_1W_2$, with $W_1 : \mathfrak{F}\oplus\mathfrak{D}_{T^*}\oplus\mathfrak{F}' \to \mathfrak{F}\oplus\mathfrak{D}_T\oplus\mathfrak{F}'$ and $W_2 : \mathfrak{F}\oplus\mathfrak{D}_T\oplus\mathfrak{F}' \to \mathfrak{F}\oplus\mathfrak{D}_{T^*}\oplus\mathfrak{F}'$ are given by the matrices*

$$W_1 = \begin{bmatrix} T & D_{T^*} & 0 \\ D_T & -T^* & 0 \\ 0 & 0 & I_{\mathfrak{F}'} \end{bmatrix}, \quad W_2 = \begin{bmatrix} I_{\mathfrak{F}} & 0 \quad 0 \\ 0 & \\ 0 & \raisebox{1ex}{Z} \end{bmatrix}.$$

1. *Every model triple can be obtained, up to unitary equivalence, in the manner described above.*
2. *The data $(\mathfrak{F}, \mathfrak{F}', T, Z)$ and $(\mathfrak{F}_1, \mathfrak{F}'_1, T_1, Z_1)$ determine equivalent model triples if and only if there exist unitary operators $A : \mathfrak{F} \to \mathfrak{F}_1$ and $A' : \mathfrak{F}' \to \mathfrak{F}'_1$ satisfying $AT = T_1A$ and*

$$((A\oplus A')|\mathfrak{D}_T\oplus\mathfrak{F}')Z = Z_1(A\oplus A')|\mathfrak{D}_{T^*}\oplus\mathfrak{F}'$$

.

The uniqueness assertion in (2) follows from the fact that the construction of $\mathfrak{F}, \mathfrak{F}', T$, and Z from $(\mathfrak{E}, U, P)$ is invariant under unitary equivalence.

It is instructive to give the explicit form of the model operators V_{U_1,P_1} and V_{U_2,P_2} in terms of the operators T and Z in Proposition 3.1. So we will take

$$\mathfrak{E} = \mathfrak{F}\oplus\mathfrak{D}_T\oplus\mathfrak{F}'$$

and consequently also

$$H^2(\mathfrak{E}) = H^2(\mathfrak{F})\oplus H^2(\mathfrak{D}_T)\oplus H^2(\mathfrak{F}').$$

The operator-valued functions appearing in the definition of V_{U_1,P_1} and V_{U_2,P_2} are

$$U_1(zP_1 + P_1^\perp) = U(zP + P^\perp)$$

and

$$U_2(zP_2 + P_2^\perp) = (P + zP^\perp)U^*,$$

respectively. In matrix form the first is

$$\begin{bmatrix} T & D_{T^*} & 0 \\ D_T & -T^* & 0 \\ 0 & 0 & I_{\mathfrak{F}'} \end{bmatrix} \begin{bmatrix} I_{\mathfrak{F}} & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{Z}} \\ 0 & & \end{bmatrix} \begin{bmatrix} I_{\mathfrak{F}} & 0 & 0 \\ 0 & \multicolumn{2}{c}{zI_{\mathfrak{E}\ominus\mathfrak{F}}} \\ 0 & & \end{bmatrix} =$$

$$= \begin{bmatrix} \mathfrak{T} & D_{\mathfrak{T}^*} & 0 \\ D_{\mathfrak{T}} & -T^* & 0 \\ 0 & 0 & I_{\mathfrak{F}'} \end{bmatrix} \begin{bmatrix} I_{\mathfrak{F}} & 0 & 0 \\ 0 & \multicolumn{2}{c}{zZ} \\ 0 & & \end{bmatrix}$$

and the second is

$$\begin{bmatrix} zI_{\mathfrak{F}} & 0 & 0 \\ 0 & \multicolumn{2}{c}{Z^*} \\ 0 & & \end{bmatrix} \begin{bmatrix} T^* & D_{\mathfrak{T}} & 0 \\ D_{\mathfrak{T}^*} & -T & 0 \\ 0 & 0 & I_{\mathfrak{F}'}. \end{bmatrix}.$$

Consequently, we have

$$\left\| V_2^* \begin{bmatrix} f \\ d \\ \varphi \end{bmatrix} \right\|^2 = \|g\|^2 + \left\| \begin{bmatrix} d \\ \varphi \end{bmatrix} \right\|^2,$$

where $g \in H^2(\mathfrak{E})$ is given by

$$g(z) = [f(z) - f(0)]/z \qquad (0 \neq z \in \mathbb{D})$$

and where

$$f \in H^2(\mathfrak{E}), d \in H^2(\mathfrak{D}_T), \qquad \varphi \in H^2(\mathfrak{F}').$$

Thus

$$\ker\, V_2^* = \mathfrak{F},$$

where $\mathfrak{F}$ *is viewed as the subspace of* $H^2(\mathfrak{E})$ *formed by the constant functions with values in* $\mathfrak{F}$. It readily follows that

$$V_1^* | \ker\, V_2^* = T^*$$

and hence

$$T = P_{\ker\, V_2^*} V_1 | \ker\, V_2^*.$$

Thus we obtain the following result.

Proposition 3.2. *The family of all operators of the form* $P_{ker\ V_2^*} V_1 | ker\ V_2^*$*, when* $\{V_1, V_2\}$ *runs over all c.n.u. bi-isometries, is (up to a unitary equivalence) the family of all contractions in Hilbert spaces.*

Continuing our study of the operators V_1, V_2, we next introduce the space $\mathfrak{F}_u$ of the unitary part of T and notice that

$$V_1 \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Tf \\ 0 \\ 0 \end{bmatrix} \qquad f \in H^2(\mathfrak{F}_u)$$

and hence

$$V_1(H^2(\mathfrak{F}_u) \oplus \{0\} \oplus \{0\}) = H^2(\mathfrak{F}) \oplus \{0\} \oplus \{0\}.$$

Thus, $H^2(\mathfrak{F}_u)$ *viewed as a subspace of* $H^2(\mathfrak{E})$, is the subspace $H^2(\mathfrak{E})_u^{(1)}$ of the unitary part of V_1 and

$$(V_1 f)(z) = Tf(z) \qquad z \in \mathbb{D}, f \in H^2(\mathfrak{F}_u).$$

Therefore, $H^2(\mathfrak{F}_u)$ reduces V_1.

On the other hand we have

$$V_2^n \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} z^n T^{*^n} f \\ 0 \\ 0 \end{bmatrix} \qquad n = 0, 1, \ldots, f \in H^2(\mathfrak{F}_u).$$

and consequently

$$V_2 H^2(\mathfrak{F}_u) \subset H^2(\mathfrak{F}_u).$$

But

$$\bigvee_{n\geq 0} V_2^n \mathfrak{F} = \mathfrak{F} \oplus V_2 \mathfrak{F} \oplus V_2^2 \mathfrak{F} \oplus \cdots$$

is the space of the c.n.u. part of V_2 and so, since

$$H^2(\mathfrak{F}_u) = \mathfrak{F}_u \oplus V_2 \mathfrak{F}_u \oplus V_2^2 \mathfrak{F}_u \oplus \cdots ,$$

it is obvious that $H^2(\mathfrak{F}_u)$ reduces V_2. We have thus obtained the following.

Lemma 3.3. *Let $\mathfrak{F}_u$ be the space of the unitary part of T. Then $H^2(\mathfrak{F}_u)$ reduces both V_1, V_2 and is included in the space $H^2(\mathfrak{E})_u^{(1)}$ of the unitary part of V_1 and in that of the c.n.u. part of V_2, i.e. $H^2(\mathfrak{E})_{cnu}^{(2)}$.*

In our further investigation of the structure of the operators V_1, V_2, we can now restrict our attention to the restrictions of V_1 and V_2 to $H^2(\mathfrak{E}) \ominus H^2(\mathfrak{F}_u)$. That means that, without loss of generality, we can assume that $\mathfrak{F}_u = \{0\}$ during this investigation. Then for $f \in \mathfrak{F}$ and $g \in H^2(\mathfrak{E})_u^{(1)}$ we have (with $\langle \cdot, \cdot \rangle$ denoting the scalar product in $H^2(\mathfrak{E})$)

$$\begin{aligned} \langle f, g \rangle = \langle f, V_1^n V_1^{*^n} g \rangle = \langle V_1^{*^n} f, V_1^{*^n} g \rangle = \\ = \langle T^{*^n} f, V_1^{*^n} g \rangle \to 0 \end{aligned}$$

if $\|T^{*^n} f\| \to 0$ for $n \to \infty$.

We will consider now the case when the latter convergence holds for all $f \in \mathfrak{F}$, that is, the case when

$$T \in C_{\cdot 0}.$$

The above calculation shows in this case that

$$\mathfrak{F} \perp H^2(\mathfrak{E})_u^{(1)}.$$

But

$$H^2(\mathfrak{E})_{cnu}^{(2)} = \bigvee_{n=0}^{\infty} V_2^n \mathfrak{F} = \mathfrak{F} \oplus V_2 \mathfrak{F} \oplus V_2^2 \mathfrak{F} \oplus \cdots$$

and for $n \geq 1$

$$\ker V_2^{*^n} = \mathfrak{F} \oplus V_2 \mathfrak{F} \oplus \cdots \oplus V_2^{n-1} \mathfrak{F}.$$

Thus

$$V_1^* \ker V_2^{*^n} \subset \ker V_2^{*^n}$$

and consequently

$$V_1^* H^2(\mathfrak{E})_{cnu}^{(2)} \subset H^2(\mathfrak{E})_{cnu}^{(2)}.$$

At this stage in our study we need the following

Lemma 3.4. *Let A on the Hilbert space $H^2(\mathfrak{G})$ be contractive commuting with the canonical shift $S_{\mathfrak{E}}$. If $A^{*^n} | \ker S_{\mathfrak{E}}^* \to 0$ strongly for $n \to \infty$, then $A^{*^n} \to 0$ strongly too.*

Proof. The operator A is the multiplication operator on $H^2(\mathfrak{G})$ given by a bounded analytic operator-valued function

$$A(z) = A_0 + zA_1 + \cdots, \quad \|A(z)\| \leq 1 \quad (z \in \mathbb{D})$$

on $\mathbb{D}$, where the A_j are operators on $\mathfrak{G}$. The condition in the statement means that $A_0^{*^n} \to 0$ strongly. For $g(z) = g_0 + zg_1 + \cdots z^N g_N$, we have that $A^{*^n} g$ has the form

$$(A^{*^n} g)(z) = g_0^{(n)} + zg_1^{(n)} + \cdots + z^{N-1} g_{N-1}^{(n)} + z^N A_0^{*^n} g_N.$$

Thus

$$\mathcal{L} = \overline{\lim_{n\to\infty}} \|A^{*^n} g\|^2 = \limsup \|A^{*^n}(g - S_{\mathfrak{E}}^N g_N)\|.$$

Iterating this argument we obtain finally that

$$\mathcal{L} = \lim \|A_0^{*^n} g_0\| = 0. \qquad \square$$

Returning to our study, we conclude that

$$V_1^{*^n} | H^2(\mathfrak{E})^{(2)}_{cnu} \to 0 \text{ strongly.}$$

The argument establishing the orthogonality $\mathfrak{F} \perp H^2(\mathfrak{E})^{(1)}_u$ now implies that

$$H^2(\mathfrak{E})^{(2)}_{cnu} \perp H^2(\mathfrak{E})^{(1)}_u.$$

Therefore

$$H^2(\mathfrak{E})^{(2)}_u = H^2(\mathfrak{E}) \ominus H^2(\mathfrak{E})^{(2)}_u \supset H^2(\mathfrak{E})^{(1)}_u.$$

But for any $h \in H^2(\mathfrak{E})^{(2)}_u$ and any $n = 1, 2, \dots$ we have

$$h = V_2^n h_n, \text{ where } h_n \in H^2(\mathfrak{E})^{(2)}_u$$

so

$$V_1^n h = (V_1 V_2)^n h_n = z^n h_n \in z^n H^2(\mathfrak{E}).$$

Therefore

$$H^2(\mathfrak{E})^{(1)}_u = V_1^n H^2(\mathfrak{E})^{(1)}_u \subset z^n H^2(\mathfrak{E})$$

for all $n = 1, 2, \dots$. This implies

$$H^2(\mathfrak{E})^{(1)}_u = \{0\}.$$

Returning to the case when $\mathfrak{F}_u$ may not be $\{0\}$, we have thus established the following structure theorem.

Theorem 3.5. *Let $T = T_{cnu} \oplus T_u$ be the canonical decomposition of T into its cnu part T_{cnu} (on $\mathfrak{F}_{cnu}$) and unitary part T_u (on $\mathfrak{F}_u$). In case*

$$T_{cnu} \in C_{\cdot 0},$$

then $H^2(\mathfrak{F})(\subset H^2(\mathfrak{E}))$ is the space of the unitary part of V_1.

This space is also reducing for V_2 and is contained in the space of the unitary part of V_2.

Recalling that any bi-isometry $\{V_1, V_2\}$ is an orthogonal sum of its model $\{V_{U_1,P_1}, V_{U_1 P_2}\}$ and a bi-isometry formed by unitary operators, we infer the following.

Corollary 3.6. *Let $\{V_1, V_2\}$ be any bi-isometry. If the c.n.u. part of*

$$V_1^* | \ker V_2^*$$

is a C_{10} contraction, then the Wold decomposition of V_1 reduces V_2 too.

Remark 3.7. 1) Let $\{V_1, V_2\}$ be a bi-isometry on $\mathfrak{H}$ such that the Wold decomposition of V_1 also reduces V_2. Let $\mathfrak{H} = \mathfrak{H}^{(1)}_{cnu} \oplus \mathfrak{H}^{(1)}_u$ be that decomposition. Thus

$$\ker V_2^* = \ker(V_2|\mathfrak{H}^{(1)}_{cnu})^* \oplus \ker(V_2|\mathfrak{H}^{(1)}_u)^*$$

and (with a little abuse of notation)

$$\begin{aligned} T^* = V_1^*|\ker V_2^* &= [(V_1^*|\mathfrak{H}^{(1)}_{cnu})/(ker(V_2|\mathfrak{H}^{(1)}_{cnu})^*)]\oplus \\ &\oplus [(V_1^*|\mathfrak{H}^{(1)}_u)/\ker(V_2^*|\mathfrak{H}^{(1)}_u)^*]. \end{aligned}$$

In the direct sum it is clear that the first operator is a $C_{0\cdot}$-contraction, while the second is unitary since $\ker(V_2|\mathfrak{H}^{(1)}_u)^*$ reduces $V_1^*|\mathfrak{H}^{(1)}_u$. Thus

$$T_{cnu} = [(V_1^*|\mathfrak{H}^{(1)}_{cnu})/\ker(V_2|\mathfrak{H}^{(1)}_{cnu})^*]^*$$

is a $C_{\cdot 0}$ contraction.

So the converse statement to the above Corollary is also valid.

2) If $\{V_1, V_2\}$ is a bi-isometry and $\dim\ker V_2^* < \infty$, then for the c.n.u. part of $T(= (V_1^*|\ker V_2^*)^*$ we have $\|T^n_{cnu}\| \to 0$. Thus the Corollary applies.

We proceed now to a more detailed analysis of the unitary operator Z. Define a contraction T' on $\mathfrak{F}'$ by $T'f' = P_{\mathfrak{F}'}Z(0 \oplus f')$, $f' \in \mathfrak{F}'$. The Julia–Halmos matrix associated with T' must again be "part" of the operator Z. More precisely, consider the decompositions $\mathfrak{D}_T \oplus \mathfrak{F}' = \mathfrak{R} \oplus (\mathfrak{F}' \vee Z^*\mathfrak{F}'$, $\mathfrak{D}_{T^*} \oplus \mathfrak{F}' = \mathfrak{R}_* \oplus (\mathfrak{F}' \vee Z\mathfrak{F}')$, $\mathfrak{D}_T = \mathfrak{R} \oplus \mathfrak{D}'_*$, and $\mathfrak{D}_{T^*} = \mathfrak{R}_* \oplus \mathfrak{D}'$, so that $\mathfrak{D}'_* \oplus \mathfrak{F}' = \mathfrak{F}' \vee Z^*\mathfrak{F}'$ and $\mathfrak{D}' \oplus \mathfrak{F}' = \mathfrak{F}' \vee Z\mathfrak{F}'$. We have $Z(\mathfrak{D}'_* \oplus \mathfrak{F}') = \mathfrak{D}' \oplus \mathfrak{F}'$ and $Z\mathfrak{R} = \mathfrak{R}_*$. As before, there exist unique unitary operators $X : \mathfrak{D}_{T'} \to \mathfrak{D}'$ and $X_* : \mathfrak{D}_{T'^*} \to \mathfrak{D}'_*$ such that

$$Z(d'_* \oplus f') = [-XT'^*X_*^*d'_* + XD_{T'}f'] \oplus [D_{T'^*}X_*^*d'_* + T'f']$$

for $d'_* \oplus f' \in \mathfrak{D}'_* \oplus \mathfrak{F}'$. In other words, Z is uniquely determined by T', X, X_*, and the unitary operator $Y = Z|\mathfrak{R} : \mathfrak{R} \to \mathfrak{R}_*$. We summarize again for future use.

Proposition 3.8. *Consider a nonet $(\mathfrak{F}, \mathfrak{F}', \mathfrak{R}, \mathfrak{R}_*, T, T', X, X_*, Y)$, where $\mathfrak{F}$ and $\mathfrak{F}'$ are Hilbert spaces, T and T' are contractions on $\mathfrak{F}$ and $\mathfrak{F}'$, respectively, $\mathfrak{R} \subset \mathfrak{D}_T$, $\mathfrak{R}_* \subset \mathfrak{D}_{T^*}$ are subspaces, and $X : \mathfrak{D}_{T'} \to \mathfrak{D}_{T^*} \ominus \mathfrak{R}_*$; and $X_* : \mathfrak{D}_{T'^*} \to \mathfrak{D}_T \ominus \mathfrak{R}$, $Y : \mathfrak{R} \to \mathfrak{R}_*$ are unitary operators. Associated with this data there is a model triple $(\mathfrak{E}, U, P)$, where $\mathfrak{E} = \mathfrak{F} \oplus \mathfrak{D}_T \oplus \mathfrak{F}'$, $P\mathfrak{E} = \mathfrak{F} \oplus \{0\} \oplus \{0\}$, and $U = W_1W_2$, with $W_1 : \mathfrak{F} \oplus \mathfrak{D}_{T^*} \oplus \mathfrak{F}' \to \mathfrak{F} \oplus \mathfrak{D}_T \oplus \mathfrak{F}'$ and $W_2 : \mathfrak{F} \oplus \mathfrak{D}_T \oplus \mathfrak{F}' \to \mathfrak{F} \oplus \mathfrak{D}_{T^*} \oplus \mathfrak{F}'$ given by the matrices*

$$W_1 = \begin{bmatrix} T & D_{T^*} & 0 \\ D_T & -T^* & 0 \\ 0 & 0 & I_{\mathfrak{F}'} \end{bmatrix}, \quad W_2 = \begin{bmatrix} I_{\mathfrak{F}} & 0 & 0 \\ 0 & & \\ 0 & Z & \end{bmatrix},$$

where $Z : \mathfrak{D}_T \oplus \mathfrak{F}' = \mathfrak{R} \oplus X_*\mathfrak{D}_{T'^*} \oplus \mathfrak{F}' \to \mathfrak{D}_{T^*} \oplus \mathfrak{F}' = \mathfrak{R}_* \oplus X\mathfrak{D}_{T'} \oplus \mathfrak{F}'$ is the unitary given by

$$Z = \begin{bmatrix} Y & 0 & 0 \\ 0 & -XT'^*X_*^* & XD_{T'} \\ 0 & D_{T'^*}X_*^* & T' \end{bmatrix}.$$

(1) *Every model triple can be obtained, up to unitary equivalence, in the manner described above.*

(2) *Two nonets* $(\mathfrak{F}_j, \mathfrak{F}'_j, \mathfrak{R}_j, \mathfrak{R}_{*j}, T_j, T'_j, X_j, X_{*j}, Y_j)$, $j = 1, 2$, *determine equivalent model triples if and only if there exist unitary operators* $A : \mathfrak{F}_1 \to \mathfrak{F}_2$ *and* $A' : \mathfrak{F}'_1 \to \mathfrak{F}'_2$ *satisfying* $AT_1 = T_2A$, $A'T'_1 = T'_2A'$, $A\mathfrak{R}_1 = \mathfrak{R}_2$, $A'\mathfrak{R}_{*1} = \mathfrak{R}_{*2}$, $(A'|\mathfrak{R}_{*1})Y_1 = Y_2(A|\mathfrak{R}_1)$, $AX_1 = X_2A'|\mathfrak{D}_{T_1}$, *and* $AX_{*1} = X_{*2}A'|\mathfrak{D}_{T_1'^*}$.
(3) *Two contractions* T *and* T' *can be included in one of the nonets described above if and only if* $\dim \mathfrak{D}_{T'} \leq \dim \mathfrak{D}_{T^*}$, $\dim \mathfrak{D}_{T'^*} \leq \dim \mathfrak{D}_T$, *and*

$$\dim \mathfrak{D}_T + \dim \mathfrak{D}_{T'} = \dim \mathfrak{D}_{T^*} + \dim \mathfrak{D}_{T'^*}.$$

Part (3) of the above statement simply gives conditions for the existence of subspaces $\mathfrak{R}$ and $\mathfrak{R}_*$ such that the various unitaries in a nonet can be constructed.

4. Irreducible Multi-Isometries

In the classification of multi-isometries, it seems natural to consider first the irreducible ones, that is, those which do not have a common reducing subspace. Assume that $(V_1, V_2, \ldots, V_n)$ is an irreducible n-isometry, and V_j is a unitary operator for some j. The spectral subspaces of V_j are hyperinvariant for V_j, hence reducing for the n-isometry. We conclude that V_j has no nontrivial spectral subspaces, so that $V_j = \lambda_j I$ for some scalar λ_j. Clearly our n-isometry will be as easy to study as the $(n-1)$-isometry obtained by deleting V_j. An n-isometry will be said to be proper if none of the component isometries is a constant multiple of the identity operator.

Lemma 4.1. *Any irreducible proper* n*-isometry* $(V_1, V_2, \ldots, V_n)$ *is cnu, and* $V_1V_2 \cdots V_n$ *is a unilateral shift of multiplicity at least* n.

Proof. The unitary part is a reducing space for the n-isometry, so it must be trivial. As noted above, none of the V_j is unitary, and therefore all the inclusions

$$\mathfrak{H} \supset V_1\mathfrak{H} \supset V_1V_2\mathfrak{H} \supset \cdots \supset V_1V_2 \cdots V_n\mathfrak{H}$$

are strict, thus proving the last assertion of the lemma. □

We will see that all the V_j must in fact be pure isometries if $(V_1, V_2, \ldots, V_n)$ is an irreducible proper isometry such that $V_1V_2 \cdots V_n$ has finite multiplicity. We need a preliminary result, which follows from Corollary 3.6.

Lemma 4.2. *Let* V_1 *and* V_2 *be commuting isometries such that* $\dim \ker V_2^* < \infty$. *Consider the von Neumann–Wold decomposition* $V_1 = S \oplus U$ *on* $\mathfrak{H} \oplus \mathfrak{H}'$ *such that* S *is pure and* U *is unitary. Then* $\mathfrak{H}$ *is a reducing subspace for* V_2.

This lemma enables us to prove the following

Corollary 4.3. *Let* F *be an irreducible family of commuting isometries such that* $\dim \ker V^* < \infty$ *for every* $V \in F$. *Then each* $V \in F$ *is either pure, or a constant multiple of the identity.*

Proof. Assume that a $V \in F$ is neither pure, nor unitary. Then the preceding lemma provides a reducing subspace for F. If V is unitary, but not a scalar multiple of the identity, then any nontrivial spectral space for V reduces F. □

Corollary 4.4. *Let* $(V_1, V_2, \ldots, V_n)$ *be an irreducible proper* n*-isometry such that* $V_1V_2 \cdots V_n$ *is a pure isometry of multiplicity* $\leq 2n-1$. *Then each* V_i *is pure, and one of them has multiplicity one. When* V_1 *has multiplicity one, we must have* $V_j = \varphi_j(V_1)$ *where* φ_j *is a finite Blaschke product for* $j = 2, 3, \ldots, n$, *and the sum*

of the multiplicities of the φ_j must be at most $2n-2$. If $V_1V_2\cdots V_n$ has multiplicity n, then each V_i must have multiplicity one.

Proof. The preceding result shows that the V_i are pure, so that the inclusions

$$\mathfrak{H} \supset V_1\mathfrak{H} \subset V_1V_2\mathfrak{H} \supset \cdots \supset V_1V_2\cdots V_n\mathfrak{H}$$

are strict, and

$$\dim[(V_1V_2\cdots V_{i-1}\mathfrak{H}) \ominus (V_1V_2\cdots V_i\mathfrak{H})] = \dim[\mathfrak{H} \ominus V_i\mathfrak{H}], \quad i = 1, 2, \ldots, n.$$

The conclusion follows because the sum of these n positive integers is $\leq 2n-1$. □

Corollary 4.5. *Let $(V_1, V_2, \cdots, V_n)$ be an irreducible proper n-isometry, and assume that $V_1V_2\cdots V_n$ has multiplicity n. Then there exist Möbius transformations φ_j such that $V_j = \varphi_j(V_1)$ for $j = 2, 3, \ldots, n$.*

Proof. The fact that $V_j = \varphi_j(V_1)$ for some $\varphi_j \in H^\infty$ follows because V_j commutes with V_1, and V_1 is a pure isometry of multiplicity one. Moreover $\varphi(V_1)$ is an isometry if and only if φ is an inner function, and the multiplicity of $\varphi(V_1)$ is the multiplicity of φ. □

The last result allows us to classify completely all irreducible proper n-isometries $(V_1, V_2, \cdots, V_n)$ for which $V_1V_2\cdots V_n$ is a shift of multiplicity n. Indeed, up to unitary equivalence, we may assume that V_1 is the shift S of multiplicity one on H^2, so that our n-isometry is $(S, \varphi_2(S), \ldots, \varphi_n(S))$ for some Möbius transforms $\varphi_2, \ldots, \varphi_n$.

Proposition 4.6. *Let $(\varphi_j)_{j=2}^n$ and $(\psi_j)_{j=2}^n$ be two families of inner functions. The n-isometries $(S, \varphi_2(S), \ldots, \varphi_n(S))$ and $(S, \psi_2(S), \ldots, \psi_n(S))$ are unitarily equivalent if and only if $\varphi_j = \psi_j$ for $j = 2, \ldots, n$.*

Proof. Let U be a unitary operator on H^2 satisfying $US = SU$ and $U\varphi_j(S) = \psi_j(S)U$. Then U must in fact be a scalar multiple of the identity, so that $\varphi_j(S) = \psi_j(S)$ and therefore $\varphi_j = \psi_j$. □

This result classifies n-isometries whenever V_1 is a shift of multiplicity one.

As mentioned in the introduction, the unitary invariants of irreducible proper n-isometries, such that $V_1V_2\cdots V_n$ has multiplicity n, can be described explicitly for $n = 2$ and $n = 3$. When $n = 2$ we must simply describe (up to unitary equivalence) all pairs (U, P), where U is unitary and P is a projection of rank one on a space $\mathfrak{E}$ of dimension 2. Using unitary equivalence, we may assume that $\mathfrak{E} = \mathbb{C}^2$, and

$$P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The possible unitary operators U are given by

$$U(c, \theta) = \begin{bmatrix} c & d\theta \\ d & \bar{c}\theta \end{bmatrix} \text{ with } |\theta| = 1, |c| < 1, \text{ and } d = (1 - |c|^2)^{1/2}.$$

Moreover, if W is a unitary operator on $\mathbb{C}^2$ such that $WP = PW$ and $WU(c, \theta) = W(c'\theta')U$, the reader will verify with no difficulty that necessarily $c = c'$ and $\theta = \theta'$. Thus (c, θ) is a complete set of unitary invariants for 2-isometries (V_1, V_2) for which the multiplicity of V_1V_2 is two.

When $n = 3$, the unitary invariants consist of commuting unitaries U_1, U_2 and projections P_1, P_2 of rank one on a space $\mathfrak{E}$ of dimension 3 satisfying $P_1 + U_1^* P_1 U_1 = P_2 + U_2^* P_2 \le I_{\mathfrak{E}}$. As before, we may assume that $\mathfrak{E} = \mathbf{C}^3$,

$$P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } U_1^* P_2 U_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Denoting by e_j the standard basis vectors in $\mathbb{C}^3$, there will exist complex numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \eta$ such that

$$U_2^* e_3 = \overline{\alpha} e_3 + \beta e_2, U_1^* U_2^* e_3 = \gamma e_3 + \delta e_1, U_1 e_2 = \varepsilon e_3 + \eta e_2.$$

Setting $N = (U_1 - \eta)(U_2^* - \overline{\alpha}) - \beta\varepsilon$, we note that N is normal and $Ne_2 = Ne_3 = 0$. If $N \ne 0$, it follows that the span of e_2 and e_3 reduces U_1, U_2, P_1, and P_2, so that the corresponding 3-isometry is reducible as well. Thus in the irreducible case we must have $N = 0$. An easy (but tedious) calculation shows that irreducibility also implies $|\alpha| < 1, |\eta| < 1$, and $\beta\varepsilon \ne 0$. The equation $N = 0$ can then be solved for U_1, yielding

$$U_1 = \theta(U_2 - \alpha)(I - \overline{\alpha} U_2)^{-1},$$

where the number $\theta = \varepsilon/\overline{\beta}$ has absolute value one. Finally, applying a unitary equivalence with a diagonal operator on $\mathbb{C}^3$, we can assume that

$$U_3 = \begin{bmatrix} \alpha_1 & \theta_1 \overline{\alpha} d_1 & -\theta_1 d d_1 \\ d_1 & -\theta_1 \overline{\alpha\alpha_1} & \theta_1 \overline{\alpha_1} d \\ 0 & d & \alpha \end{bmatrix},$$

where $d = (1 - |\alpha|^2)^{1/2}, d_1 = (1 - |\alpha_1|^2)^{1/2}$, and $|\theta_1| = 1$. We must also have $|\alpha_1| < 1$ on account of irreducibility. Given numbers $\alpha, \alpha_1, \theta, \theta_1$ such that $|\alpha| < 1$, $|\alpha_1| < 1$, and $|\theta| = |\theta_1| = 1$, let us denote by $U_1(\alpha, \alpha_1, \theta, \theta_1)$ and $U_2(\alpha, \alpha_1, \theta, \theta_1)$ the unitary operators given by the above formulas. In this way the quadruple $(\alpha, \alpha_1, \theta, \theta_1)$ determines a 3-isometry (V_1, V_2, V_3) such that the multiplicity of $V_1 V_2 V_3$ is three. Again, the reader will be able to verify that a unitary W satisfying $WP_j = P_j W$ and $WU_1(\alpha, \alpha_1, \theta, \theta_1) = U_1(\alpha', \alpha_1', \theta', \theta_1')W$ exists only in case $\alpha = \alpha'$, $\alpha_1 = \alpha_1'$, $\theta = \theta'$, and $\theta_1 = \theta_1'$.

It is interesting to note that, in case $V_1 V_2 \cdots V_n$ is a shift of multiplicity at least $2n$, the isometries V_j need not all belong to the algebra generated by some shift of multiplicity one. A simple example is obtained for $n = 2$ with $V_1 = S \oplus S$, and

$$V_2 = \begin{bmatrix} 0 & S^2 \\ I & 0 \end{bmatrix},$$

where S denotes the standard shift of multiplicity one on H^2. For this example we have $V_1^2 = V_2^2$, so that $(V_1 - V_2)(V_1 + V_2) = 0$, while $V_1 - V_2 \ne 0 \ne V_1 + V_2$. The commutant of a shift of multiplicity one is isomorphic to H^∞, and this algebra has no zero divisors; thus V_1 and V_2 cannot belong to the commutant of the same shift of multiplicity one.

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